Intuitionistic fuzzy rough sets model based on (Θ, Φ)-operators

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***Abstract*—The extension of rough set model is an important research direction in rough set theory. The aim of this paper is to present a new extension. At the first, we introduce a pair of dual intuitionistic fuzzy operators (Θ, Φ). And some important properties are examined about these these operators. Moreover, θ-lower and ϕ-upper approximation operators are defined, by using the operators, and a novel intuitionistic fuzzy rough set model is constructed based on an intuitionistic fuzzy equivalence relation. Furthermore, some valuable properties are obtained in the model.**

***Index Terms*—Approximation operators; Intuitionistic fuzzy relation; Intuitionistic fuzzy rough sets; Triangular norm.**

I. Introduction

Rough set theory, proposed by Pawlak [8], [9], is a theory for the research of uncertainty management in a wide variety of applications related to artificial intelligence [6]. The theory has been applied successfully in the fields of pattern recognition, medical diagnosis, data mining, conflict analysis, algebra [11], which related an amount of imprecise, vague and uncertain information.

Atanassov [1] presented intuitionistic fuzzy (IF, briefly) set in 1986 which is very effective to deal with vagueness. As a generalization of fuzzy set [13], the concept of IF set has played an important role in the analysis of uncertainty of data [5], [7], [12]. Combining IF set theory and rough set theory may result in a new hybrid mathematical structure for the requirement of knowledge-handling systems. In recent years, Various definitions of IF rough set were explored to extend rough set theory in the IF environment [4], [10].

The purpose of this paper is to investigate intuitionistic fuzzy rough set model based on a pair of dual intuitionistic fuzzy implicators, i.e. Φ-upper and Θ-lower approximation operators defined on the basis of these two implicators. The rest of this paper is organized as follows. Some preliminary concepts of IF sets and two IF implications are showed in Section 2. In Section 3, we propose the concepts and operations of IF rough sets and discuss their properties. Finally, in Section 4, we draw the conclusion.

II. Preliminaries

In this section, we introduce some basic notions and properties related to IF sets. We first review a special lattice on $I^2 = [0,1]^2$ originated by [3].

Definition 2.1([3]) Let $L^* = \{(\alpha_1, \alpha_2) \in I^2 | 0 \le \alpha_1 + \alpha_2 \le 1\}$. The order relation $\le_{L^*}$ on L^* is defined as follows: for all $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in L^*$,

$(\alpha_1, \alpha_2) \le_{L^*} (\beta_1, \beta_2) \Leftrightarrow \alpha_1 \le \beta_1$ and $\alpha_2 \ge \beta_2$.

Then the relation $\le_{L^*}$ is a partial ordering on L^* and the pair $(L^*, \le_{L^*})$ is a complete lattice with the smallest element $0_{L^*} = (0, 1)$ and the greatest element $1_{L^*} = (1, 0)$. The meet operator $\wedge$, join operator $\vee$ and complement operator $\sim$ on $(L^*, \le_{L^*})$ which are linked to the ordering $\le_{L^*}$ are, respectively, defined as follows: for all $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in L^*$,

$(\alpha_1, \alpha_2) \wedge (\beta_1, \beta_2) = (\min(\alpha_1, \beta_1), \max(\alpha_2, \beta_2))$,

$(\alpha_1, \alpha_2) \vee (\beta_1, \beta_2) = (\max(\alpha_1, \beta_1), \min(\alpha_2, \beta_2))$.

$\sim (\alpha_1, \alpha_2) = (\alpha_2, \alpha_1)$.

Meanwhile the order relation $\ge_{L^*}$ on L^* is defined as follows: for all $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in L^*$,

$(\beta_1, \beta_2) \ge_{L^*} (\alpha_1, \alpha_2) \Leftrightarrow (\alpha_1, \alpha_2) \le_{L^*} (\beta_1, \beta_2)$,

$\alpha = \beta \Leftrightarrow \alpha \le_{L^*} \beta$ and $\beta \le_{L^*} \alpha \Leftrightarrow \alpha_1 = \beta_1, \alpha_2 = \beta_2$,

$\alpha <_{L^*} \beta \Leftrightarrow \alpha \le_{L^*} \beta$ and $\alpha \ne \beta$.

Definition 2.2([1]) Let a set U be fixed. An IF set A on U is an object having the form

$$A = \{\langle x, \mu_A(x), \nu_A(x) \rangle | x \in U\},$$

where $\mu_A : U \to I$ and $\nu_A : U \to I$ satisfy $0 \le \mu_A(x) + \nu_A(x) \le 1$ for all $x \in U$; $\mu_A(x)$ and $\nu_A(x)$ are called the degree of membership and the degree of non-membership of the element $x \in U$ to A, respectively. The family of all IF subsets of U is denoted by $IF(U)$. The complement of an IF set A is defined by $\sim A = \{\langle x, \nu_A(x), \mu_A(x) \rangle | x \in U\}$.

Obviously, every fuzzy set $A = \{\langle x, \mu_A(x) \rangle | x \in U\}$ can be identified with the IF set of the form $A = \{\langle x, \mu_A(x), 1 - \mu_A(x) \rangle | x \in U\}$. We denote the family of all fuzzy subsets on U as $F(U)$.

Next, we introduce some basic operations on $IF(U)$ as follows.

Definition 2.3([1]) If $A, B \in IF(U)$, then,

(1)$A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$ for all $x \in U$,

(2)$A \supseteq B \Leftrightarrow B \subseteq A$,

(3)$A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$,

(4)$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)) | x \in U \rangle\}$,

(5) $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))|x \in U\rangle\}$.

For $\alpha = (\alpha_1, \alpha_2) \in L^*$, $\widehat{\alpha} = \widehat{(\alpha_1, \alpha_2)}$ will be denoted by the constant IF set: $\widehat{\alpha}(x) = \widehat{(\alpha_1, \alpha_2)}(x) = (\alpha_1, \alpha_2)$, for all $x \in U$. In particularly, if $a \in I$ we denote $\widehat{a}$ as a constant fuzzy set, i.e., $\widehat{a}(x) = a$ for all $x \in U$.

For any $y \in U$, IF set $\widehat{1_y}$ and $\widehat{1_{U-\{y\}}}$ are, respectively, define as follows: for all $x \in U$,

$$\mu_{\widehat{1_{\{y\}}}}(x) = \begin{cases} 1, & \text{if } x = y, \\ 0. & \text{if } x \neq y. \end{cases}$$

$$\nu_{\widehat{1_{\{y\}}}}(x) = \begin{cases} 0, & \text{if } x = y, \\ 1. & \text{if } x \neq y. \end{cases}$$

$$\mu_{\widehat{1_{U-\{y\}}}}(x) = \begin{cases} 0, & \text{if } x = y, \\ 1. & \text{if } x \neq y. \end{cases}$$

$$\nu_{\widehat{1_{U-\{y\}}}}(x) = \begin{cases} 1, & \text{if } x = y, \\ 0. & \text{if } x \neq y. \end{cases}$$

The IF universe set is $\widehat{1_U} = \widehat{(1,0)} = \widehat{1_{L^*}} = \{\langle x, 1, 0\rangle | x \in U\}$ and the IF empty set is $\widehat{1_\emptyset} = \widehat{(0,1)} = \widehat{0_{L^*}} = \{\langle x, 0, 1\rangle | x \in U\}$.

Definition 2.4([14]) A fuzzy triangular norm (briefly fuzzy t-norm) on I is an increasing, commutative, associative mapping $T : I \times I \to I$ satisfying $T(1, a) = a$ for all $a \in I$.

A fuzzy t-conorm (briefly fuzzy t-conorm) on I is an increasing, commutative, associative mapping $S : I \times I \to I$ satisfying $T(0, a) = a$ for all $a \in I$.

A fuzzy t-norm T and a fuzzy t-conorm S on I are said to be dual with respect to complement operator $\sim$, if for all $a, b \in I$,

$S(a, b) =\sim T(\sim a, \sim b) = 1 - T(1 - a, 1 - b)$.

Definition 2.5([3]) An IF t-norm $\mathcal{T}$(respectively, t-conorm $\mathcal{S}$) on L^* can be defined by fuzzy t-norm T and t-conorm S as follows: for all $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in L^*$

$\mathcal{T}(\alpha, \beta) = (T(\alpha_1, \beta_1), S(\alpha_2, \beta_2))$

$\mathcal{S}(\alpha, \beta) = (S^{'}(\alpha_1, \beta_1), T^{'}(\alpha_2, \beta_2))$.

Definition 2.6 Let T be a fuzzy t-norm on I and S be the dual of T. For any $a, b, c \in I$, two fuzzy residual implication by the T and S are defined as:

$\theta(a, b) = \sup\{c \in I | T(a, c) \leq b\}$,

$\phi(a, b) = \inf\{c \in I | S(a, c) \geq b\}$.

Now, we define the following two IF implication on L^*: for all $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in L^*$,

$\Phi(\alpha, \beta) = (\phi(1 - \alpha_2, \beta_1), \theta(1 - \alpha_1, \beta_2))$,

$\Theta(\alpha, \beta) = (\theta(1 - \alpha_2, \beta_1), \phi(1 - \alpha_1, \beta_2))$.

Proposition 2.1 Let θ be a fuzzy residual implication and ϕ be the dual of θ, for any $a, b \in I$, then $\phi(\sim a, \sim b) =\sim \theta(a, b)$.

Proof: By the definition of θ and ϕ, we have

$$\begin{aligned} \phi(\sim a, \sim b) &= \inf\{c \in I | S(\sim a, c) \geq \sim b\} \\ &= \inf\{c \in I | \sim T(a, \sim c) \geq \sim b\} \\ &= \inf\{\sim d \in I | T(a, d) \leq b\} \\ &=\sim \sup\{d \in I | T(a, d) \leq b\} \\ &=\sim \theta(a, b). \end{aligned}$$

■

Obviously, it can be seen that $\Phi(\alpha, \beta) =\sim \Theta(\sim \alpha, \sim \beta)$, for all $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in L^*$.

Proposition 2.2 The binary operation ϕ and θ enjoy the following properties: $\forall a, b, c \in I$

(1) $\phi(0, a) = a, \phi(1, a) = 0,\ \phi(a, 0) = 0$;

(2) $a \leq b \Leftrightarrow \phi(c, a) \leq \phi(c, b)$;

(3) $a \leq b \Leftrightarrow \phi(a, c) \geq \phi(b, c)$;

(4) $a \geq b \Leftrightarrow \phi(a, b) = 0$;

(5) $\phi(\bigwedge\limits_i a_i, \bigvee\limits_j b_j) = \bigvee\limits_i \bigvee\limits_j \phi(a_i, b_j)$;

(6) $\phi(\bigvee\limits_i a_i, \bigwedge\limits_j b_j) = \bigwedge\limits_i \bigwedge\limits_j \phi(a_i, b_j)$;

(7) $\bigvee\limits_{a \in I} \phi(\phi(b, a), a) = b$;

(8) $\phi(a, \phi(b, c)) = \phi(b, \phi(a, c))$;

(9) $\phi(S(a, b), c)) = \phi(a, \phi(b, c))$;

(10) $a \leq \phi(b, c) \Leftrightarrow b \leq \phi(a, c)$

and

(1′) $\theta(1, a) = a, \theta(0, a) = 1,\ \theta(a, 1) = 1$;

(2′) $a \leq b \Leftrightarrow \theta(c, a) \leq \theta(c, b)$;

(3′) $a \leq b \Leftrightarrow \theta(a, c) \geq \theta(b, c)$;

(4′) $a \leq b \Leftrightarrow \theta(a, b) = 1$;

(5′) $\theta(\bigwedge\limits_i a_i, \bigvee\limits_j b_j) = \bigvee\limits_i \bigvee\limits_j \theta(a_i, b_j)$;

(6′) $\theta(\bigvee\limits_i a_i, \bigwedge\limits_j b_j) = \bigwedge\limits_i \bigwedge\limits_j \theta(a_i, b_j)$;

(7′) $\bigwedge\limits_{a \in I} \theta(\theta(b, a), a) = b$;

(8′) $\theta(a, \theta(b, c)) = \theta(b, \theta(a, c))$;

(9′) $\theta(T(a, b), c)) = \theta(a, \theta(b, c))$;

(10′) $a \geq \theta(b, c) \Leftrightarrow b \geq \theta(a, c)$.

Proof: The Proposition can be easily proved by Definition 2.6. ■

III. Construction of (Θ, Φ)-IF rough set

In this section, by using two IF implication Θ and Φ, we introduce the concept of IF rough set and investigate some properties of IF rough approximation operators.

Here, we first recall the concept of IF $\mathcal{T}$ equivalence relation.

Definition 3.1([2]) An IF binary relation R on U is an IF subset of $U \times U$, namely, R is given by

$$R = \{\langle (x, y), \mu_R(x, y), \nu_R(x, y)\rangle | (x, y) \in U \times U\},$$

where $\mu_R : U \times U \to I$ and $\nu_R : U \times U \to I$, $0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1$ for all $(x, y) \in U \times U$. $IFR(U \times U)$ will be used to denote the family of all IF relations on U.

Definition 3.2([2]) Let $R \in IFR(U \times U)$, we say that

(1) R is referred to as a reflexive IF relation if for any $x \in U$, $R(x, x) = 1$.

(2) R is referred to as a symmetric IF relation if for any $x, y \in U$, $R(x, y) = R(y, x)$.

(3) R is referred to as a $\mathcal{T}$ transitive IF relation if for any $x, y, z \in U$, $R(x, z) \geq_{L^*} \mathcal{T}(R(x, y), R(y, z))$.

If R is reflexive, symmetric and $\mathcal{T}$ transitive on U, then we say that R is an IF $\mathcal{T}$ equivalence relation on U.

Definition 3.3 Let U be a finite nonempty universe of discourse, and R be an IF relation on U. The pair (U, R) is called a generalized IF approximation space. The Φ-upper and Θ-lower approximations of a set $A \in IF(U)$ with respect to an IF relation R are respectively defined by

$\overline{R}(A) = \{\langle x, \mu_{\overline{R}(A)}(x), \nu_{\overline{R}(A)}(x)\rangle | x \in U\}$,

$\underline{R}(A)=\{\langle x,\mu_{\underline{R}(A)}(x),\nu_{\underline{R}(A)}(x)\rangle|x\in U\}$,
where

$\mu_{\overline{R}(A)}(x)=\underset{y\in U}{\vee}\phi(1-\mu_R(x,y),\mu_A(y))$,

$\nu_{\overline{R}(A)}(x)=\underset{y\in U}{\wedge}\theta(1-\nu_R(x,y),\nu_A(y))$;

$\mu_{\underline{R}(A)}(x)=\underset{y\in U}{\wedge}\theta(1-\nu_R(x,y),\mu_A(y))$,

$\nu_{\underline{R}(A)}(x)=\underset{y\in U}{\vee}\phi(1-\mu_R(x,y),\nu_A(y))$.

The pair $(\underline{R},\overline{R})$ is called the (Θ,Φ)-IF rough set of A with respect to (U,R). Let R be an IF $\mathcal{T}$ equivalence relation on U. The pair (U,R) is called an IF approximation space.

The Φ-upper and Θ-lower approximations of a set $A\in IF(U)$ with respect to an IF equivalence relation R can be expressed as: for all $x\in U$,

$\underline{R}(A)(x)=\underset{y\in U}{\wedge}\Theta(R(x,y),A(y))$;

$\overline{R}(A)(x)=\underset{y\in U}{\vee}\Phi(\sim R(x,y),A(y))$.

Let $A\in IF(U)$ and $R\in IF(U\times U)$, $\forall x\in U$, we have $\nu_A(x)\leq 1-\mu_A(x)$ and $\nu_R(x,y)\leq 1-\mu_R(x,y)$, then

$$\begin{aligned}\mu_{\overline{R}(A)}(x)&=\underset{y\in U}{\vee}\phi(1-\mu_R(x,y),\mu_A(y))\\&=1-\underset{y\in U}{\wedge}\theta(\mu_R(x,y),1-\mu_A(y))\\&\leq 1-\underset{y\in U}{\wedge}\theta(1-\nu_R(x,y),\nu_A(y))\\&=1-\nu_{\overline{R}(A)}(x),\end{aligned}$$

so $\mu_{\overline{R}(A)}(x)+\nu_{\overline{R}(A)}(x)\leq 1$. Thus we have proved that $\overline{R}(A)\in IF(U)$. Similarly, we can verify that $\underline{R}(A)\in IF(U)$. Based on this conclusion, we call $\underline{R}$, $\overline{R}:IF(U)\to IF(U)$ the Θ-lower and Φ-upper IF rough approximation operators, respectively.

If $\overline{R}(A)\neq\underline{R}(A)$, then the IF set A is an IF rough set on the IF $\mathcal{T}$ equivalence relation.

Remark 3.1 Another natural definitions of the Φ-upper and Θ-lower approximations of a set $A\in IF(U)$ with respect to an IF $\mathcal{T}$ equivalence relation R could be defined by:

$\overline{R}(A)=\{\langle x,\mu_{\overline{R}(A)}(x),\nu_{\overline{R}(A)}(x)\rangle|x\in U\}$,

$\underline{R}(A)=\{\langle x,\mu_{\underline{R}(A)}(x),\nu_{\underline{R}(A)}(x)\rangle|x\in U\}$;

where

$\mu_{\overline{R}(A)}(x)=\underset{y\in U}{\vee}\phi(\nu_R(x,y),\mu_A(y))$,

$\nu_{\overline{R}(A)}(x)=\underset{y\in U}{\wedge}\theta(\mu_R(x,y),\nu_A(y))$;

$\mu_{\underline{R}(A)}(x)=\underset{y\in U}{\wedge}\theta(\mu_R(x,y),\mu_A(y))$,

$\nu_{\underline{R}(A)}(x)=\underset{y\in U}{\vee}\phi(\nu_R(x,y),\nu_A(y))$.

However, we can verify that $\overline{R}(A)\in IF(U)$ and $\underline{R}(A)\in IF(U)$ are not true by the following example.

Example 3.1 let $U=\{x_1,x_2,x_3\}$, $A=\{(0.6,0.3),(0.3,0.5),(0.9,0.1)\}$, and

$$R=\begin{pmatrix}(1,0)&(0.88,0.08)&(0.88,0.08)\\(0.88,0.08)&(1,0)&(1,0)\\(0.88,0.08)&(1,0)&(1,0)\end{pmatrix}.$$

We assume the IF t-norm $\mathcal{T}$ as: $\mathcal{T}(\widehat{\alpha},\widehat{\beta})=(T(\alpha_1,\beta_1),S(\alpha_2,\beta_2))$, where $\widehat{\alpha}=(\alpha_1,\alpha_1),\widehat{\beta}=(\beta_1,\beta_2)$, $T(\alpha_1,\beta_1)=\max\{0,\alpha_1+\beta_1-1\}$, $S(\alpha_2,\beta_2)=\min\{1,\alpha_2+\beta_2\}$.

It can be found that R is an IF $\mathcal{T}$ equivalence relation on U. And we can calculate the $\overline{R}(A)$ as follows:

$\mu_{\overline{R}(A)}(x_1)=\phi(0,0.6)\vee\phi(0.08,0.3)\vee\phi(0.08,0.9)=0.82$,

$\mu_{\overline{R}(A)}(x_2)=\phi(0.08,0.6)\vee\phi(0,0.3)\vee\phi(0,0.9)=0.9$,

$\mu_{\overline{R}(A)}(x_3)=\phi(0.08,0.6)\vee\phi(0,0.3)\vee\phi(0,0.9)=0.9$;

$\nu_{\overline{R}(A)}(x_1)=\theta(1,0.3)\wedge\theta(0.88,0.5)\wedge\theta(0.88,0.1)=0.22$,

$\nu_{\overline{R}(A)}(x_2)=\theta(0.88,0.3)\wedge\theta(1,0.5)\wedge\theta(1,0.1)=0.1$,

$\nu_{\overline{R}(A)}(x_3)=\theta(0.88,0.3)\wedge\theta(1,0.5)\wedge\theta(1,0.1)=0.1$.

Then $\overline{R}(A)=\{(0.82,0.22),(0.9,0.1),(0.9,0.1)\}$, Obviously $\overline{R}(A)\notin IF(U)$.

Theorem 3.1 Let (U,R) be an IF approximation space, $\underline{R}$ and $\overline{R}$ are the Θ-lower and Φ-upper IF rough approximation operators defined in Definition 3.3. $\forall A,B\in IF(U)$, $\alpha=(\alpha_1,\alpha_2)\in L^*$, Then

(1) $\underline{R}(\sim A)=\sim\overline{R}(A)$, $\overline{R}(\sim A)=\sim\underline{R}(A)$.

(2) $\underline{R}(A)\subseteq A\subseteq\overline{R}(A)$.

(3) $\underline{R}(A\cap B)=\underline{R}(A)\cap\underline{R}(B)$, $\overline{R}(A\cup B)=\overline{R}(A)\cup\overline{R}(B)$.

(4) $A\subseteq B\Rightarrow\underline{R}(A)\subseteq\underline{R}(B)$ and $\overline{R}(A)\subseteq\overline{R}(B)$.

(5) $\underline{R}(A\cup B)\supseteq\underline{R}(A)\cup\underline{R}(B)$, $\overline{R}(A\cap B)\subseteq\overline{R}(A)\cup\overline{R}(Y)$.

(6) $\underline{R}(\widehat{\alpha})=\widehat{\alpha}$, $\overline{R}(\widehat{\alpha})=\widehat{\alpha}$.

In particular, $\underline{R}(\emptyset)=\overline{R}(\emptyset)=\emptyset$, $\underline{R}(U)=\overline{R}(U)=U$.

(7) $\underline{R}(\underline{R}(A))=\underline{R}(A)$, $\overline{R}(\overline{R}(A))=\overline{R}(A)$.

Proof: (1) From Definition 3.3 and Proposition 2.1, $\forall x\in U$ we have

$$\begin{aligned}\mu_{\underline{R}(\sim A)}(x)&=\underset{y\in U}{\wedge}\theta(1-\nu_R(x,y),\mu_{(\sim A)}(y))\\&=\underset{y\in U}{\wedge}\theta(1-\nu_R(x,y),\nu_A(y))=\nu_{\overline{R}(A)}(x),\\\nu_{\underline{R}(\sim A)}(x)&=\underset{y\in U}{\vee}\phi(1-\mu_R(x,y),\nu_{(\sim A)}(y))\\&=\underset{y\in U}{\vee}\phi(1-\mu_R(x,y),\mu_A(y))=\mu_{\overline{R}(A)}(x).\end{aligned}$$

Thus, $\underline{R}(\sim A)=\sim\overline{R}(A)$,

Similarly, we can obtain that $\overline{R}(\sim A)=\sim\underline{R}(A)$.

(2) $\forall x\in U$, we have

$$\begin{aligned}\mu_{\underline{R}(A)}(x)&=\underset{y\in U}{\wedge}\theta(1-\nu_R(x,y),\mu_A(y))\\&\leq\theta(1-\nu_R(x,x),\mu_A(x))\\&=\theta(1,\mu_A(x))=\mu_A(x)\\\nu_{\underline{R}(A)}(x)&=\underset{y\in U}{\vee}\phi(1-\mu_R(x,y),\nu_A(y))\\&\geq\phi(1-\mu_R(x,x),\nu_A(x))\\&=\phi(0,\nu_A(x))=\nu_A(x).\end{aligned}$$

Thus, $\underline{R}(A)\subseteq A$.

$A\subseteq\overline{R}(A)$ follows immediately from conclusion $\underline{R}(A)\subseteq A$ and the dual properties.

(3) $\forall x\in U$, we have

$$\begin{aligned}&\mu_{\underline{R}(A\cap B)}(x)=\underset{y\in U}{\wedge}\theta(1-\nu_R(x,y),\mu_{(A\cap B)}(y))\\&=\underset{y\in U}{\wedge}\theta(1-\nu_R(x,y),\mu_{(A)}(y)\wedge\mu_{(B)}(y))\\&=[\underset{y\in U}{\wedge}\theta(1-\nu_R(x,y),\mu_A(y))]\wedge[\underset{y\in U}{\wedge}\theta(1-\nu_R(x,y),\mu_B(y))]\\&=\mu_{\underline{R}(A)}(x)\wedge\mu_{\underline{R}(B)}(x),\end{aligned}$$

$$\nu_{\underline{R}(A\cap B)}(x) = \bigvee_{y\in U} \phi(1-\mu_R(x,y), \nu_{(A\cap B)}(y))$$
$$= \bigvee_{y\in U} \phi(1-\mu_R(x,y), \nu_{(A)}(y) \vee \nu_{(B)}(y))$$
$$=[\bigvee_{y\in U} \phi(1-\mu_R(x,y), \nu_A(y))] \vee [\bigvee_{y\in U} \phi(1-\mu_R(x,y), \nu_B(y))]$$
$$=\nu_{\underline{R}(A)}(x) \vee \nu_{\underline{R}(B)}(x).$$

Then $\underline{R}(A\cap B) = \underline{R}(A) \cap \underline{R}(B)$.

Similarly, we can get that $\overline{R}(A\cup B) = \overline{R}(A) \cup \overline{R}(B)$.

(4) Since $A \subseteq B$, i.e., $\mu_A(x) \le \mu_B(x)$, $\nu_B(x) \le \nu_A(x)$ for all $x \in U$, we have

$$\begin{aligned}\mu_{\underline{R}(A)}(x) &= \bigwedge_{y\in U} \theta(1-\nu_R(x,y), \mu_A(y)) \\ &\le \bigwedge_{y\in U} \theta(1-\nu_R(x,y), \mu_B(y)) = \mu_{\underline{R}(B)}(x), \\ \nu_{\underline{R}(A)}(x) &= \bigvee_{y\in U} \phi(1-\mu_R(x,y), \nu_A(y)) \\ &\ge \bigvee_{y\in U} \phi(1-\mu_R(x,y), \nu_B(y)) = \nu_{\underline{R}(B)}(x)\end{aligned}$$

Then $\underline{R}(A) \subseteq \underline{R}(B)$.

Similarly, we can acquire that $\overline{R}(A) = \overline{R}(B)$.

(5) It follows immediately from (4).

(6) Since $\forall x \in U$,$\widehat{\alpha}(x) = \alpha = (\alpha_1, \alpha_2)$, then we have

$$\begin{aligned}\mu_{\underline{R}}(\widehat{\alpha})(x) &= \bigwedge_{y\in U} \theta(1-\nu_R(x,y), \mu_{\widehat{\alpha}}(y)) \\ &= \bigwedge_{y\in U} \theta(1-\nu_R(x,y), \alpha_1) \\ &= \theta(\bigvee_{y\in U} (1-\nu_R(x,y)), \alpha_1) = \theta(1, \alpha_1) = \alpha_1, \\ \nu_{\underline{R}}(\widehat{\alpha})(x) &= \bigvee_{y\in U} \phi(1-\mu_R(x,y), \nu_{\widehat{\alpha}}(y)) \\ &= \bigvee_{y\in U} \phi(1-\mu_R(x,y), \alpha_2) \\ &= \phi(\bigwedge_{y\in U} (1-\nu_R(x,y)), \alpha_2) = \phi(0, \alpha_2) = \alpha_2.\end{aligned}$$

Thus $\underline{R}(\widehat{\alpha}) = \widehat{\alpha}$.

Similarly, we can achieve that $\overline{R}(\widehat{\alpha}) = \widehat{\alpha}$.

Take $\widehat{\alpha} = \emptyset$ in the above proof, then we have $\underline{R}(\emptyset) = \overline{R}(\emptyset) = \emptyset$, take $\widehat{\alpha} = U$, we get $\underline{R}(U) = \overline{R}(U) = U$.

(7) By (2), we can easily know that $\underline{R}(\underline{R}(A)) \subseteq \underline{R}(A)$ and $\underline{R}(A) \subseteq \overline{R}(\underline{R}(A))$. $\forall x \in U$, we have

$$\begin{aligned}\mu_{\underline{R}(\underline{R}(A))}(x) &= \bigwedge_{y\in U} \theta(1-\nu_R(x,y), \mu_{\underline{R}(A)}(y)) \\ &= \bigwedge_{y\in U} \theta(1-\nu_R(x,y), \bigwedge_{z\in U} \theta(1-\nu_R(y,z), \mu_A(z))) \\ &= \bigwedge_{y\in U}\bigwedge_{z\in U} \theta(1-\nu_R(x,y),\, \theta(1-\nu_R(y,z), \mu_A(z))) \\ &= \bigwedge_{y\in U}\bigwedge_{z\in U} \theta(T(1-\nu_R(x,y), 1-\nu_R(y,z)),\, \mu_A(z)) \\ &\ge \bigwedge_{z\in U} \theta(1-\nu_R(x,z),\, \mu_A(z)) = \mu_{\underline{R}(A)}(x), \\ \nu_{\underline{R}(\underline{R}(A))}(x) &= \bigvee_{y\in U} \phi(1-\mu_R(x,y), \nu_{\underline{R}(A)}(y)) \\ &= \bigvee_{y\in U} \phi(1-\mu_R(x,y), \bigvee_{z\in U} \phi(1-\mu_R(y,z), \nu_A(z))) \\ &= \bigvee_{y\in U}\bigvee_{z\in U} \phi(1-\mu_R(x,y),\, \phi(1-\mu_R(y,z), \nu_A(z))) \\ &= \bigvee_{y\in U}\bigvee_{z\in U} \phi(S(1-\mu_R(x,y), 1-\mu_R(y,z)),\, \nu_A(z)) \\ &\le \bigvee_{z\in U} \phi(1-\mu_R(x,z),\, \nu_A(z)) = \nu_{\underline{R}(A)}(x)\end{aligned}$$

So that, $\underline{R}(\underline{R}(A)) \supseteq \underline{R}(A)$. Thus $\underline{R}(\underline{R}(A)) = \underline{R}(A)$.

Moreover, the formula $\overline{R}(\overline{R}(A)) = \overline{R}(A)$ can be examined immediately from conclusion $\underline{R}(\underline{R}(A)) = \underline{R}(A)$ and the dual properties. ■

We observe from Theorem 3.1(7) that $\underline{R}(\underline{R}(A)) = \underline{R}(A)$, $\overline{R}(\overline{R}(A)) = \overline{R}(A)$. But $\overline{R}(\underline{R}(A)) = \underline{R}(A)$ and $\underline{R}(\overline{R}(A)) = \overline{R}(A)$ are not hold.

Example 3.2 Consider the IF approximation space of Example 3.1. we can calculate $\underline{R}(\overline{R}(A))$ and $\overline{R}(A)$ as follows:

$\mu_{\overline{R}(A)}(x_1) = \phi(0, 0.6) \vee \phi(0.12, 0.3) \vee \phi(0.12, 0.9) = 0.78$,
$\mu_{\overline{R}(A)}(x_2) = \phi(0.12, 0.6) \vee \phi(0, 0.3) \vee \phi(0, 0.9) = 0.9$,
$\mu_{\overline{R}(A)}(x_3) = \phi(0.12, 0.6) \vee \phi(0, 0.3) \vee \phi(0, 0.9) = 0.9$;
$\mu_{\underline{R}(\mu_{\overline{R}(A)})}(x_1) = \theta(1, 0.78) \wedge \theta(0.92, 0.9) \wedge \theta(0.92, 0.9) = 0.78$,
$\mu_{\underline{R}(\mu_{\overline{R}(A)})}(x_2) = \theta(0.92, 0.78) \wedge \theta(1, 0.9) \wedge \theta(1, 0.9) = 0.86$,
$\mu_{\underline{R}(\mu_{\overline{R}(A)})}(x_3) = \theta(0.92, 0.78) \wedge \theta(1, 0.9) \wedge \theta(1, 0.9) = 0.86$.

Therefore, $\underline{R}(\overline{R}(A)) \ne \overline{R}(A)$.

Theorem 3.2 Let (U, R) and (U, S) be two IF approximation spaces, $S \subseteq R$ and $A \in IF(U)$, then $\underline{R}(A) \subseteq \underline{S}(A)$, $\overline{S}(A) \subseteq \overline{R}(A)$.

Proof: Since $S \subseteq R$, i.e., $\mu_S(x,y) \le \mu_R(x,y)$, $\nu_R(x,y) \le \nu_S(x,y)$, for all $x, y \in U$, then $\forall x \in U$, we have

$$\begin{aligned}\mu_{\underline{R}(A)}(x) &= \bigwedge_{y\in U} \theta(1-\nu_R(x,y), \mu_A(y)) \\ &\le \bigwedge_{y\in U} \theta(1-\nu_S(x,y), \mu_A(y)) = \mu_{\underline{S}(A)}(x), \\ \nu_{\underline{R}(A)}(x) &= \bigvee_{y\in U} \phi(1-\mu_R(x,y), \nu_A(y)) \\ &\ge \bigvee_{y\in U} \phi(1-\mu_S(x,y), \nu_A(y)) = \nu_{\underline{S}(A)}(x).\end{aligned}$$

Thus $\underline{R}(A) \subseteq \underline{S}(A)$.

On the other hand, $\overline{S}(A) \subseteq \overline{R}(A)$ follows immediately from $\underline{R}(A) \subseteq \underline{S}(A)$ and the dual properties. ■

Theorem 3.3 Let (U, R) be an IF approximation space, for any $x, y \in U$, $\alpha = (\alpha_1, \alpha_2) \in L^*$.

$\underline{R}(\Theta(1_{\widehat{\{y\}}}, \widehat{\alpha}))(x) = \underline{R}(\Theta(1_{\widehat{\{x\}}}, \widehat{\alpha}))(y) = \Theta(R(x,y), \alpha)$,

i.e.,

$\mu_{\underline{R}(\Theta(1_{\widehat{\{y\}}}, \widehat{\alpha}))}(x) = \mu_{\underline{R}(\Theta(1_{\widehat{\{y\}}}, \widehat{\alpha}))}(x)$
$= \theta(1-\nu_R(x,y), \alpha_1)$
$\nu_{\underline{R}(\Theta(1_{\widehat{\{y\}}}, \widehat{\alpha}))}(x) = \nu_{\underline{R}(\Theta(1_{\widehat{\{y\}}}, \widehat{\alpha}))}(x)$
$= \phi(1-\mu_R(x,y), \alpha_2)$

$\overline{R}(\Phi(1_{\widehat{U-\{y\}}}, \widehat{\alpha}))(x) = \overline{R}(\Phi(1_{\widehat{U-\{x\}}}, \widehat{\alpha}))(y)$
$= \Phi(\sim R(x,y), \alpha)$,

i.e.,

$\mu_{\overline{R}(\Phi(1_{\widehat{U-\{y\}}}, \widehat{\alpha}))}(x) = \mu_{\overline{R}(\Phi(1_{U-\{x\}}, \widehat{\alpha}))}(y)$
$= \phi(1-\mu_R(x,y), \alpha_1)$,
$\nu_{\overline{R}(\Phi(1_{\widehat{U-\{y\}}}, \widehat{\alpha}))}(x) = \nu_{\overline{R}(\Phi(1_{U-\{x\}}, \widehat{\alpha}))}(y)$
$= \theta(1-\nu_R(x,y), \alpha_2)$.

Proof: From the definition of $\overline{R}$ and Proposition 2.2, we have

$$\begin{aligned}
&\mu_{\overline{R}(\Phi(\widehat{1_{U-\{y\}}},\widehat{\alpha}))}(x)\\
&= \bigvee_{z\in U} \phi(1-\mu_R(x,z), \mu_{\phi(\widehat{\mu_{1_{U-\{y\}}}},\widehat{\alpha_1})}(z))\\
&= \phi(1-\mu_R(x,y), \mu_{\phi(\widehat{\mu_{1_{U-\{y\}}}},\widehat{\alpha_1})}(y))\\
&\quad \vee (\bigvee_{z\neq y} \phi(1-\mu_R(x,z), \mu_{\phi(\widehat{\mu_{1_{U-\{y\}}}},\widehat{\alpha_1})}(z)))\\
&= \phi(1-\mu_R(x,y),\alpha_1) \vee (\bigvee_{z\neq y} \phi(1-\mu_R(x,z),0))\\
&= \phi(1-\mu_R(x,y),\alpha_1),\\
&\nu_{\overline{R}(\Phi(\widehat{1_{U-\{y\}}},\widehat{\alpha}))}(x)\\
&= \bigwedge_{z\in U} \theta(1-\nu_R(x,z), \nu_{\theta(\widehat{\nu_{1_{U-\{y\}}}},\widehat{\alpha_2})}(z))\\
&= \theta(1-\nu_R(x,y), \nu_{\theta(\widehat{\nu_{1_{U-\{y\}}}},\widehat{\alpha_2})}(y))\\
&\quad \wedge (\bigwedge_{z\neq y} \theta(1-\nu_R(x,z), \nu_{\theta(\widehat{\nu_{1_{U-\{y\}}}},\widehat{\alpha_2})}(z)))\\
&= \theta(1-\nu_R(x,y),\alpha_2) \wedge (\bigwedge_{z\neq y} \theta(1-\nu_R(x,z),1))\\
&= \theta(1-\nu_R(x,y),\alpha_2).
\end{aligned}$$

Then, $\overline{R}(\Phi(\widehat{1_{U-\{y\}}},\widehat{\alpha}))(x) = \Phi(\sim R(x,y),\alpha)$. By the symmetry of R, it is immediately to obtain $\overline{R}(\Phi(\widehat{1_{U-\{x\}}},\widehat{\alpha}))(y) = \Phi(\sim R(x,y),\alpha)$. Thus $\overline{R}(\Phi(\widehat{1_{U-\{y\}}},\widehat{\alpha}))(x) = \Phi(\sim R(x,y),\alpha)= \overline{R}(\Phi(\widehat{1_{U-\{x\}}},\widehat{\alpha}))(y)$.

And $\underline{R}(\Theta(\widehat{1_{\{y\}}},\widehat{\alpha}))(x) = \underline{R}(\Theta(\widehat{1_{\{x\}}},\widehat{\alpha}))(y) = \Theta(R(x,y),\alpha)$ can be got direct from the above conclusion and the dual properties. ■

Theorem 3.4 Let (U,R) be an IF approximation space, for any $\alpha=(\alpha_1,\alpha_2)\in L^*$. $\underline{R}(\Theta(\widehat{\alpha},A)) = \Theta(\widehat{\alpha},\underline{R}(A))$, i.e.,

$\mu_{\underline{R}(\Theta(\widehat{\alpha},A))} = \theta(1-\alpha_2,\mu_{\underline{R}(A)})$,

$\nu_{\underline{R}(\Theta(\widehat{\alpha},A))} = \phi(1-\alpha_1,\nu_{\underline{R}(A)})$.

$\overline{R}(\Phi(\widehat{\alpha},A)) = \Phi(\widehat{\alpha},\overline{R}(A))$, i.e.,

$\mu_{\overline{R}(\Phi(\widehat{\alpha},A))} = \phi(1-\alpha_2,\mu_{\overline{R}(A)})$,

$\nu_{\overline{R}(\Phi(\widehat{\alpha},A))} = \theta(1-\alpha_1,\nu_{\overline{R}(A)})$.

Proof: From the definition of $\overline{R}$ and Proposition 2.2, for any $x\in U$, we have

$$\begin{aligned}
\mu_{\overline{R}(\Phi(\widehat{\alpha},A))} &= \bigvee_{y\in U} \phi(1-\mu_R(x,y), \mu_{\Phi(\widehat{\alpha},A)}(y))\\
&= \bigvee_{y\in U} \phi(1-\mu_R(x,y), \phi(1-\alpha_2,\mu_A)(y))\\
&= \phi(1-\alpha_2, \bigvee_{y\in U} \phi(1-\mu_R(x,y),\mu_A)(y))\\
&= \phi(1-\alpha_2,\mu_{\overline{R}(A)}),\\
\nu_{\underline{R}(\Theta(\widehat{\alpha},A))} &= \bigwedge_{y\in U} \theta(1-\nu_R(x,y), \nu_{\Theta(\widehat{\alpha},A)}(y))\\
&= \bigwedge_{y\in U} \theta(1-\nu_R(x,y), \theta(1-\alpha_1,\nu_A)(y))\\
&= \theta(1-\alpha_1, \bigwedge_{y\in U} \theta(1-\nu_R(x,y),\nu_A)(y))\\
&= \theta(1-\alpha_1,\nu_{\underline{R}(A)}),
\end{aligned}$$

Then, $\overline{R}(\Phi(\widehat{\alpha},A)) = \Phi(\widehat{\alpha},\overline{R}(A))$.

$\underline{R}(\Theta(\widehat{\alpha},A)) = \Theta(\widehat{\alpha},\underline{R}(A))$, follows immediately from the above conclusion and the dual properties. ■

IV. Conclusion

In this paper, we defined the intuitionistic fuzzy rough sets by the Φ-upper and Θ-lower approximation operators, which is a natural extension of fuzzy rough sets. And some main properties of the intuitionistic fuzzy rough approximation operators had been given. Meanwhile, we have proved another natural extension of fuzzy rough sets is unreasonable in this paper. In further research, we will study the axiomatic approach of intuitionistic fuzzy rough sets.

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